A "CLASSICAL" GENERAL EQUILIBRIUM MODEL*

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ABSTRACT

It is shown in this paper how for any parametrically given rate of profits ρ, a price vector and an allocation exist such that: (a) Consumers maximize their preferences subject to their budget constraints; (b) Firms maximize profits; (c) All active firms are equally profitable (with a rate of return equal to ρ); and (d) All markets clear.

1. INTRODUCTION

The purpose of this paper is to formulate a consistent "classical" general equilibrium model of a competitive market economy. By consistent we mean that there is a well-defined equilibrium notion and that the set of equilibria is nonempty. By a general equilibrium model we refer to a model in which prices and quantities are interdependent and simultaneously determined, given consumers' preferences, initial endowments and technology. A competitive market economy is intended to mean a private ownership market economy where agents are price-takers. Finally, the term "classical" alludes to the notion of competition common among classical economists, such as Smith, Ricardo or Marx (but also Walras); according to this notion, competition tends to equalize profitability across industries and firms.

It is well known that most of the classical economists neglected the analysis of the demand side of the economy. Consumption was essentially conceived as part of the reproduction necessities of the economic system; wages were kept at a minimum level, just allowing for the subsistence of workers (either by some population law or by the role played by the "reservation army"). Their analysis focused on the production side, concentrating on reproduction, distribution and growth. Assuming constant returns to scale, prices were understood as "prices of production", including profits. Profits constituted the reason of the existence and activity of firms. Competition was then characterized as a situation in which free entry (which may also be interpreted as free access to the available technology) yielded the equalization of profits per unit of costs across firms and industries.

Neoclassical equilibrium models provide a proper way of modeling not only the demand side of the economy, but also the interplay between production, demand and prices. Yet, when we assume constant returns to scale (what many authors consider as the paradigm), equilibrium profits must be zero. It is then difficult to explain why firms exist or produce.

Our model tries to combine the classical notion of competition with the neoclassical approach to general equilibrium[1]. We shall concentrate on a <u>single period</u>, <u>constant returns to scale</u> economy, in order to provide insights on the nature and role of profits in a production economy. Following von Neumann's (1937) approach, the set of technological possibilities will be modeled in terms of a given number of (linear) transformations. The description of the technology includes all efficient production processes.

Agents behaviour is modeled as follows. Consumers own the initial endowments and maximize utility at given prices. A part of every

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[1] It is worth stressing in this respect that we are not trying to contribute to the history of economic thought, but to develop a model within the stream of the present economic theory.

consumer's optimal decision is to allocate wealth between consumption and ownership (which gives rise to a process of firms creation, provided consumers find it profitable). Firms may be understood as a coalition of consumers agreeing to operate some of the available technological possibilities, while, at the same time, providing the necessary means of production. Firms maximize profits at given prices, subject to their feasibility constraints. Profits are distributed among consumers in proportion to their property.

This kind of behaviour implies a clear distinction between production possibilities (which belong to the data of the model) and firms (which constitute part of consumers optimal decisions). We depart in this point from the standard Arrow-Debreu-MacKenzie general equilibrium model, since neither the number of firms nor consumers' participation in firms' profits are taken as given.

The main result is the following: for any parametrically given rate of profits ρ, a price vector and an allocation exist such that: (a) Consumers maximize their preferences subject to their budget constraints; (b) Firms maximize profits; (c) All active firms are equally profitable (with a rate of return equal to ρ); and (d) All markets clear.

This result may be interpreted as saying that, within the limits of constant returns to scale economies, our model encompasses the standard general equilibrium model (where $\rho = 0$ is the only possible equilibrium rate of return), Sraffa's (1960) model of prices of production (where ρ is also taken as an external parameter), and von Neumann's (1937) model of an expanding economy (where ρ is endogenously determined by holding constant the prices of primary factors -*labour*-).

Section 2 develops the conceptual background of the model (an informal account of what consumers and firms are, and how they behave), while Section 3 contains the formal model. Section 4 is devoted to the proof of the existence theorem. Further comments and remarks are gathered in the final Section.

2. CONSUMERS AND FIRMS IN A SINGLE PERIOD, CONSTANT RETURNS TO SCALE, PRIVATE OWNERSHIP, COMPETITIVE ECONOMY

The idea of formulating a "classical" general equilibrium model will be specified in terms of the following target: to construct a general equilibrium model of a *single period, constant returns to scale, private ownership, competitive economy*, whose equilibria may be compatible with the existence of positive profits. The interest of such a compatibility comes from the fact that, faced with a lack of profits, it is difficult to explain why firms exist or produce in a market economy.

By a general equilibrium model we refer to a model of a market economy, whose data can be summarized by a set of consumers (characterized by their consumption sets and their preferences), a set of technological production possibilities, and a vector of initial endowments.

In order to isolate the relationship between production activities and profits, it will be assumed that everything happens in *a single period* of time, and that there is no uncertainty whatsoever (that is, we concentrate on a timeless world).

Constant returns to scale refer to a technological feature consisting of the possibility of modifying arbitrarily the scale of operation of the available processes (this will be modeled as a given number of technological linear transformations, so that we implicitly assume limited substitution possibilities).

By a *private ownership economy* we mean that in this economy consumers own the initial endowments and are entitled to use the available technology for production purposes (by creating firms and appropriating the profits thereby generated).

By a *competitive economy* we refer to an economy where agents' decisions are characterized by the following features:

(a) *Price-taking behaviour:* Consumers and firms are price-takers, so that each agent thinks she can realize whatever transaction she wishes (within her feasible set), at market prices.

(b)*Free-access to technology:* Any available production process can be operated by a firm (that is, firms do not face any technological barrier).

(c) *Free trading:* Consumers are free to trade in any market; in particular, they can freely create firms (provided technological requirements are satisfied).

In a market economy, firms are the institutions that give consumers (who own all "means of production") the opportunity of exploiting the benefits derived from the division of labour[2]. Therefore firms will appear as part of the optimal decisions of consumers, and will actually be created only when consumers find it individually profitable for them to do so (profits appear then as the natural market solution for this incentive problem).

Firms may be regarded as free licenses to use certain technology. Then, any set of agents can "create" a firm by making available those means of production that technology stipulates for the operation of the corresponding production processes. Therefore, each firm is owned by those consumers supplying the required means of production. Property is divided in proportion to the value of consumers' contributions.

Consumers maximize utility by suitably choosing consumption bundles under the restriction of their available wealth. Wealth is obtained from the exchange value of their initial endowments. Hence any consumer has to decide how to allocate her endowments between the available options: trading at market prices with the consumption sector, or participating in the property of some firms (according to their profitability).

A *firm* is, then, a coalition of consumers agreeing to operate some of the available technological possibilities while, at the same time,

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2 Observe that if technology is productive enough, society might be better-off in a production economy; this is so because production allows to change the nature, size and composition of the initial endowments according to consumers' preferences.

providing the required means of production to enable this. Since this is part of consumers' rational behaviour, under competitive conditions (and assuming that consumers are non-satiable) firms will maximize profits which will be divided in proportion to consumers' property shares. Observe that constant returns to scale and competitive conditions allow consumers to compute firms profitability as a rate of return (that is, in terms of profits per unit of wealth "invested"), and that no consumer will willingly accept to participate in a firm whose profitability be less than the maximum she can get (both because she can always leave a coalition and because she can always join a new one, at no cost). Profits constitute the signal that consumers are willing to create firms as a way of reallocating the resources.

We shall distinguish between producible commodities and primary inputs (or <u>factors</u>). In this timeless world, producible commodities operate as a necessary "catalyst" for production activities, whilst primary factors are actually used up. Therefore, the amounts of primary inputs available constitute the relevant restrictions for the productive sectors.

The operation of this constant returns to scale private ownership competitive economy can be described by the following logical sequence.

1.- First the auctioneer calls a price vector. This determines each technological process profitability (profits per unit of costs).

2.- Consumers take the given price vector and the rates of return associated to each production process and decide their consumption and the allocation of their initial endowments. The said allocation involves a decision on participating in coalitions (or firms property) in order to exploit the benefits obtainable by operating the most profitable production processes.

3.- The firms created in this way, will operate those production processes yielding the highest rates of return, and will select those activity levels which maximize profits[3].

4.- Then the auctioneer collects all those messages and checks out whether they are compatible or not. In the first case an equilibrium is obtained, whilst in the second one a new iteration begins.

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[3] Notice that positive profits are compatible with profit maximizing firms. This is so because every coalition of consumers (that is, every possible firm) has a limited amount of factors: at most the sum of every participant's initial endowments. Hence, for each price vector, every firm's production set is truncated by the total primary inputs provided by the consumers participating in it.

3. THE FORMAL MODEL AND THE MAIN RESULT

Consider a constant returns to scale, private ownership competitive economy with *n* producible commodities and *k* primary inputs (or "factors"). A point $\omega \in \mathbb{R}^{n+k}$ denotes the aggregate vector of initial endowments.

Following von Neumann's (1937) approach, technology is described by a list of *h* production processes. Joint production is the rule. A point $x \in \mathbb{R}^h_+$ stands for a vector of *activity levels*. Then, the *jth* production process can intuitively be described as follows:

$$D_j \quad \& \quad G_j \quad \text{---->} \quad H_j$$

where $D_j \in \mathbb{R}^n_+$, $G_j \in \mathbb{R}^k_+$ are column vectors describing the producible inputs and primary factors, respectively, required to produce $H_j \in \mathbb{R}^n_+$.

The *Technology* can be summarized by means of three matrices,

$$[\ \mathbf{H},\ \mathbf{D},\ \mathbf{G}\ ]$$

whose elements, H_{ij}, D_{ij}, and G_{tj}, respectively (with $i = 1, 2, \dots, n$, $j = 1, 2, \dots, h$, and $t = 1, 2, \dots, k$) correspond to the (average) technical coefficients of production. The description of the technology includes all available efficient production processes (that is, all relevant alternative techniques).

In order to get a compact representation of the technology, we shall denote by **B** the $(n+k) \times h$ matrix, whose first *n* rows consist of matrix **H**, and the remaining *k* correspond to a null matrix. We shall denote by **A** the

$(n+k) \times h$ matrix whose first n rows consist of matrix **D** and the remaining correspond to matrix **G**. That is,

$$\mathbf{B} \equiv \begin{bmatrix} \mathbf{H} \\ 0 \end{bmatrix} \quad , \quad \mathbf{A} \equiv \begin{bmatrix} \mathbf{D} \\ \mathbf{G} \end{bmatrix}$$

Let $\mathbf{x} \in \mathbb{R}^h_+$ be a vector of activity levels. We shall use the following expressions:

$$\mathbf{y}_j(\mathbf{x}) \equiv [\mathrm{B}_j - \mathrm{A}_j]\, \mathrm{x}_j$$

which denotes the net output generated by the operation of the *jth* process, and

$$\mathbf{y}(\mathbf{x}) \equiv (\mathbf{B} - \mathbf{A})\, \mathbf{x}$$

which denotes the vector of net outputs for the whole economy corresponding to **x**. Observe that while net outputs will appear as positive numbers, net inputs will do so as negative ones. Furthermore, if there are several firms operating a given process, we must interpret x_j as the sum of all incumbent firms' activity levels.

We shall denote by $\mathbb{P} \subset \mathbb{R}^{n+k}_+$ the standard price simplex, that is,

$$\mathbb{P} = \{\ \mathbf{p} \in \mathbb{R}^{n+k}_+ \ /\ \sum_{t=1}^{n+k} \mathrm{p}_t = 1\ \}$$

For a vector of activity levels, $\mathbf{x} \in \mathbb{R}^h_+$, and a price vector $\mathbf{p} \in \mathbb{P}$, the expression:

$$\mathbf{p}\ (\mathrm{B}_j - \mathrm{A}_j)\ \mathrm{x}_j$$

gives us the profits generated by the *jth* process.

Let $\mathbf{p} \in \mathbb{P}$ be a given price vector; then, the *jth* process *rate of profits*, $r_j(\mathbf{p})$, is given by:

$$r_j(\mathbf{p}) = \frac{\mathbf{p}\,(B_j - A_j)}{\mathbf{p}\,A_j}$$

(whenever defined).

Let us consider a mapping $\mathbf{r}:\mathbb{P} \dashrightarrow \mathbb{R}$, given by:

$$\mathbf{r}(\mathbf{p}) = \mathrm{Max}_j \{ r_j(\mathbf{p}) \}$$

This mapping (whenever defined) associates to each price vector the maximum rate of profits a process can yield.

Concerning technology we shall assume:

A.1.- $(\mathbf{H} - \mathbf{D})\mathbf{x} > 0 \implies \mathbf{A}\mathbf{x} \gg 0, \quad \forall\, \mathbf{x} \in \mathbb{R}^h_+$.

This assumption states that there cannot be positive net production (of producible commodities), without using up some amount of every commodity. This is a technological requirement which implies both the boundedness of the set of attainable allocations (when consumption sets are bounded from below, as it will be assumed below), and the "indecomposability" of the productive system [see Bidard (1991, Ch. XI) for a discussion].

There are *m* consumers. Each consumer i = 1,2,..., m, is characterized by a tuple,

$$[C_i,\ u_i,\ \omega_i\ ]$$

where C_i, u_i and ω_i stand for the *ith* consumer consumption set, utility function and initial endowments, respectively. By construction, $\sum_{i=1}^{m} \omega_i = \omega$.

Given a price vector $\mathbf{p} \in \mathbb{P}$, the *ith* consumer's behaviour is obtained by solving the following program:

$$\begin{aligned} &\text{Max.} u_i\ (c_i) \\ &\text{s.t:} \\ &c_i \in \beta_i(\mathbf{p}) \end{aligned}$$

where $\beta_i(\mathbf{p})$ denotes the *ith* consumer's budget set at prices $\mathbf{p}$ (to be specified later on). The mapping $\mathbf{d}_i : \mathbb{P} \dashrightarrow C_i$ associating with every price vector the set of points c_i solving the program above, will be called the *ith* consumer's demand. The aggregate net demand will then be given by:

$$\xi(\mathbf{p}) = \mathbf{d}(\mathbf{p}) - \omega$$

where

$$\mathbf{d}(\mathbf{p}) = \sum_{i=1}^{m} \mathbf{d}_i(\mathbf{p})$$
$$\omega = \sum_{i=1}^{m} \omega_i$$

Concerning consumers, we shall assume:

A.2.- For each consumer $i = 1,2,\dots, m$:

(i) $C_i \subset \mathbb{R}_+^{n+k}$ is a closed and convex set.

(ii) $u_i : C_i \dashrightarrow \mathbb{R}$ is a continuous and quasi-concave utility function.

(iii) $\omega_i \in C_i$, and there exists $c'_i \in C_i$ such that $c'_i << \omega_i$.

(iv) For each $c_i \in C_i$, and for every $\varepsilon > 0$, there exists c''_i in $\mathcal{B}(c_i, \varepsilon) \cap C_i$ such that, $u_i(c''_i) > u_i(c_i)$ (where $\mathcal{B}(c_i, \varepsilon)$ stands for a closed ball with center c_i and radius ε).

Assumption (A.2) is rather standard and needs little comment. Notice that we are taking all consumption bundles to be nonnegative [see Arrow & Hahn (1971, Ch. 4) for a discussion]. Then, (i) and (iii) of assumption (A.2) imply that $\omega_i >> 0$ for every i.

A firm is a coalition of consumers agreeing to operate certain processes at certain activity levels, while providing the necessary "means of production". Thus, let M = { 1, 2, ..., m } stand for the set of indices identifying the *m* consumers, and let $\mathcal{M}$ stand for the family of all possible coalitions, where we allow every consumer to participate in an arbitrary number of them. Let *g* denote an element of $\mathcal{M}$ (that is, a firm).

For each consumer *i*, every $\mathbf{p} \in \mathbb{P}$, let $\delta_i(\mathbf{p}) \in \mathbb{R}^{n+k}$ denote those amounts of initial endowments traded with the consumption sector (which includes herself), and $\psi_{ig}(\mathbf{p}) \in \mathbb{R}^{n+k}$ those devoted to productive activities, *via* the *gth* firm. By definition, $\forall\ \mathbf{p} \in \mathbb{P}$,

$$\delta_i(\mathbf{p}) + \sum_g \psi_{ig}(\mathbf{p}) = \omega_i$$

whilst the the *ith* consumer's participation in the property of the *gth* firm is given by:

$$\theta_{ig}(\mathbf{p}) = \frac{\mathbf{p}\ \psi_{ig}(\mathbf{p})}{\mathbf{p} \sum_{i \in g} \psi_{ig}(\mathbf{p})}$$

The *gth* firm can be characterized by a correspondence, $\mu_g : \mathbb{P} \dashrightarrow \mathbb{R}^k_+$, given by:

$$\mu_g(\mathbf{p}) = \sum_{i \in g} \psi^G_{ig}(\mathbf{p})$$

where $\psi^G_{ig}(\mathbf{p})$ denotes the projection of $\psi_{ig}(\mathbf{p})$ on the factors' space. Thus, $\mu_g(\mathbf{p})$ specifies the amounts of primary inputs that consumers in coalition *g* make available at prices $\mathbf{p}$.

Then the *gth* firm's supply is obtained by solving the following program:

$$\text{Max.} \sum_{j=1}^{h} \mathbf{p}\,(B_j - A_j)\, x_{gj}$$

s.t.:

$$\sum_{j=1}^{h} G_j\, x_{gj} \leq \mu_g(\mathbf{p})$$

$$x_{gj} \geq 0$$

We shall denote by $x_{gj}(\mathbf{p})$ the solution to this program, and $x_j(\mathbf{p}) \equiv \sum_g x_{gj}(\mathbf{p})$.

We are now ready to define the *ith* consumer's budget set, $\beta_i(\mathbf{p})$, which is given by:

$$\beta_i(\mathbf{p}) \equiv \{\, c_i \in C_i \;/\; \mathbf{p}c_i \leq \mathbf{p}\omega_i + \sum_g \theta_{ig}(\mathbf{p})\, \mathbf{p}\,(B_j - A_j)x_{jg}(\mathbf{p}) \,\}$$

Let $\rho \in \mathbb{R}$ be a parametrically given scalar (to be interpreted as a uniform rate of profits), and consider now the following definition:

Definition 1.- For a given $\rho \in \mathbb{R}$, we shall say that a price vector, $\mathbf{p}^* \in \mathbb{P}$, and a vector of activity levels, $\mathbf{x}^* \in \mathbb{R}^h_+$, yield a Competitive Equilibrium relative to ρ, if the following conditions hold:

(α) Every consumer i = 1,2,...,m chooses a consumption bundle, $c_i^* \in C_i$ and a distribution of her initial endowments, $\delta_i(\mathbf{p}^*)$, $\psi_{ig}(\mathbf{p}^*)$ (j = 1,2,..., h), such that $u_i(c_i^*)$ is maximum over the set of points in $\beta_i(\mathbf{p}^*)$.

(β) Every firm chooses a vector of activity levels $\mathbf{x}^*_g = (x^*_{g1}, \ldots, x^*_{gh})$, satisfying the following conditions:

β,1) $\mathbf{x}^*_g$ solves the following program:

$$\text{Max.} \sum_{j=1}^{h} \mathbf{p}^* (B_j - A_j)\, x_{gj}$$

s.t.:

$$\sum_{j=1}^{h} G_j\, x_{gj} \leq \mu_g(\mathbf{p}^*)$$

$$x_{gj} \geq 0$$

β,2) $\mathbf{x}^* \neq 0 \implies \mathbf{r}(\mathbf{p}^*) = \rho$

β,3) $\sum_{g} x^*_{gj} = x^*_j$

(γ) $\sum_{i=1}^{m} c^*_i - \omega \leq \sum_{j=1}^{h} \mathbf{y}_j(\mathbf{x}^*)$, and

$$\sum_{i=1}^{m} c^*_{is} - \omega_s < \sum_{j=1}^{h} \mathbf{y}_{js}(\mathbf{x}^*) \implies p^*_s = 0$$

That is, a Competitive Equilibrium relative to ρ is a situation in which: (a) Consumers maximize their preferences subject to their budget constraints; (b) Firms maximize profits; (c) All active firms are equally profitable (with a rate of return equal to ρ); and (d) All markets clear.

Let $\mathbb{E}$ stand for the class of economies described above [constant returns to scale private ownership competitive economies satisfying assumptions (A.1) and (A.2)]. The main result of this paper is the following:

THEOREM .- Let E **be an economy in** $\mathbb{E}$**. For any given** $\rho \in \mathbb{R}$ **there exists a Competitive Equilibrium relative to** ρ**.**

(The proof is given in the next Section)

This result ensures that under standard conditions, an equilibrium which equalizes firms' profitability exists. The equilibrium rate of profits may be positive under constant returns to scale, indicating that consumers are willing to reallocate resources (transforming primary factors into producible commodities) by engaging production activities.

Consider now the following definition, which tries to capture the basic idea of competitive equilibrium in von Neumann's model and classical economists' thinking:

Definition 2.- We shall say that $(\mathbf{p}^{**}, \mathbf{x}^{**}, \rho^*)$ yields a **Classical Competitive Equilibrium,** if the following two conditions are satisfied:

(i) $(\mathbf{p}^{**}, \mathbf{x}^{**})$ yields Competitive Equilibrium relative to ρ^*.

(ii) There is no $\rho \in \mathbb{R}$ yielding a Competitive Equilibrium relative to ρ with higher total profits.

A Classical Competitive Equilibrium is a situation in which: (a) Consumers maximize their preferences; (b) Firms maximize profits; (c) all (active) firms are equally profitable, and there is no $(\mathbf{p}', \mathbf{x}', \rho')$ such that:

$$\mathbf{p}' (\mathbf{B} - \mathbf{A}) \mathbf{x}' > \mathbf{p}^* (\mathbf{B} - \mathbf{A}) \mathbf{x}^*$$

for $(\mathbf{p}', \mathbf{x}')$ yielding a Competitive Equilibrium relative to ρ'; and (d) All markets clear.

Now, by noticing that the set of attainable aggregate profits, $\rho\, \mathbf{pAx}$, is compact and varies continuously with $(\mathbf{p}, \mathbf{x}, \rho)$, the following result turns out immediate:

Corollary.- Let E be an economy in $\mathbb{E}$. Then, a Classical Competitive Equilibrium exists.

4. THE EXISTENCE OF COMPETITIVE EQUILIBRIA

Let $\mathbb{E}$ stand for the class of economies described in Section III, that is, private ownership market economies satisfying assumptions (A.1) and (A.2). This Section is devoted to proving that for any economy in $\mathbb{E}$, a Classical Competitive Equilibrium exists. The proof will be developed in two steps. In the first one, it will be considered the existence of a solution to a suitable complementarity problem. Next, it will be shown that such a solution actually corresponds to a Competitive Equilibrium.

For each given $\mathbf{p} \in \mathbb{P}$ and every consumer, define $\alpha_i : \mathbb{P} \dashrightarrow [0, 1]$ as the *ith* consumer's share in total wealth at prices $\mathbf{p}$), that is,

$$\alpha_i(\mathbf{p}) \equiv \frac{\mathbf{p}\ \omega_i}{\mathbf{p}\ \omega}$$

It is easy to see that, under assumption (A.2), α_i is continuous (since ω is a strictly positive vector).

Let now $\hat{E}$ denote an economy whose basic data (commodities, consumers, production possibilities and available resources) coincide with those in E, but where for every $\mathbf{p} \in \mathbb{P}$, $\mathbf{x} \in \mathbb{R}^n_+$, the *ith* consumer's behaviour is obtained from solving the following program:

$$\begin{aligned} &\text{Max.} \quad u_i(c_i) \\ \text{s.t.:}& \\ &\mathbf{p}c_i \le \mathbf{p}\omega_i + \alpha_i(\mathbf{p})\mathbf{p}(\mathbf{B} - \mathbf{A})\mathbf{x} \end{aligned}$$

Let $\hat{\mathbf{d}}_i(\mathbf{p}, \mathbf{x})$ stand for the set of solutions to this program, for i = 1,2,..., m. Then, define

$$\hat{\xi}(\mathbf{p}, \mathbf{x}) = \sum_{i=1}^{m} \hat{\mathbf{d}}_i(\mathbf{p}, \mathbf{x}) - \sum_{i=1}^{m} \omega_i$$

Let $\{(c_i), [\mathbf{y}_j(\mathbf{x})]\}$ (for i = 1, 2, ..., m, j = 1, 2, ..., h) stand for an allocation. We shall denote by $X(\omega) \subset \mathbb{R}^h_+$ the set of attainable activity levels, that is, the set of points $\mathbf{x} \in \mathbb{R}^h_+$ such that:

$$\sum_{i=1}^{m} c_i - \sum_{j=1}^{h} \mathbf{y}_j(\mathbf{x}) \leq \omega$$

Under assumptions (A.1) and (A.2) $X(\omega)$ is a nonempty compact set. Define now the following sets:

$$\mathbb{X}(k) = \{ \mathbf{x} \in \mathbb{R}^h_+ \ / \ \sum_{j=1}^{h} x_j \leq k \}$$

where k is a positive scalar big enough so that $X(\omega) \subset \mathbb{X}(k)$, and:

$$\{ \mathbf{x} \in \mathbb{R}^h_+ \ / \ \sum_{j=1}^{h} x_j = k \} \cap X(\omega) = \varnothing$$

$$\Delta \equiv \mathbb{P} \times \mathbb{X}(k)$$

Define now a mapping $\Gamma_\rho: \Delta \dashrightarrow \mathbb{R}^{n+k+h}$ as follows:

$$\Gamma_\rho(\mathbf{p}, \mathbf{x}) \equiv \begin{bmatrix} \hat{\xi}(\mathbf{p}, \mathbf{x}) - \mathbf{y}(\mathbf{x}) \\ \mathbf{p}[\mathbf{B} - (1+\rho)\mathbf{A}] \end{bmatrix}$$

[where $\mathbf{y}(\mathbf{x}) \equiv (\mathbf{B} - \mathbf{A})\, \mathbf{x}$].

Consider now the following complementarity problem: Find $(\mathbf{p}^*, \mathbf{x}^*)$ in Δ, $(\mathbf{z}^*, \mathbf{v}^*) \in \Gamma_\rho(\mathbf{p}^*, \mathbf{x}^*)$, such that,

$$\left.\begin{array}{r}(z^*,\ v^*) \leq 0 \\ (p^*,\ x^*)\ (z^*,\ v^*) = 0\end{array}\right\} \qquad [1]$$

We shall show now that this complementarity problem has a solution (p^*, x^*), and that this solution corresponds to a Competitive Equilibrium relative to ρ. To do this, let us begin with the following Lemma:

Lemma.- Let D be a compact and convex subset of $\mathbb{R}^n$, and $\Psi: D \to \mathbb{R}^n$ an upper-hemicontinuous correspondence, with nonempty, compact and convex values. Then there exist points $x^* \in D$, $y^* \in \Psi(x^*)$ such that, $(x - x^*)\, y^* \leq 0$, for all $x \in D$.

[A proof of this result is given in Villar (1991, Ch. 2)]

Proposition.- Under assumptions (A.1) and (A.2), the complementarity problem [1] has a solution $(p^*, x^*) \in \Delta$.

Proof.-

Observe that assumption (A.2) implies that for all $i = 1,2,\ldots, m$, the mapping $\gamma_i: \Delta \to C_i$ given by:

$$\gamma_i(p,\ x) = \{\ c_i \in C_i \ \ / \ \ pc_i \leq p\omega_i + \alpha_i(p)p(B - A)x\ \}$$

is continuous in (p, x) [Debreu (1959, 4.8 (1))], with nonempty, compact and convex values. Therefore, since preferences are assumed to be continuous and convex, for each point (p, x) in Δ, every $\hat{d}_i(p, x)$ will be an upper-hemicontinuous correspondence (the maximum theorem applies here), with nonempty, compact and convex values. Consequently, $\hat{\xi}$ inherits these properties and, by construction and in view of assumption (A.1), this also applies for Γ_ρ.

Thus, Γ_ρ is an upper-hemicontinuous correspondence, with nonempty, compact and convex values, applying a compact and convex set $\Delta \subset \mathbb{R}_+^{n+k+h}$ into $\mathbb{R}^{n+k+h}$. Then, the Lemma ensures the existence of points $(\mathbf{p}^*, \mathbf{x}^*) \in \Delta$, $(\mathbf{z}^*, \mathbf{v}^*) \in \Gamma_\rho(\mathbf{p}^*, \mathbf{x}^*)$ such that,

$$(\mathbf{p}^*, \mathbf{x}^*)\ (\mathbf{z}^*, \mathbf{v}^*) \geq (\mathbf{p}, \mathbf{x})\ (\mathbf{z}^*, \mathbf{v}^*)$$

for every pair $(\mathbf{p}, \mathbf{x})$ in Δ. In particular, for every $\mathbf{p} \in \mathbb{P}$,

$$(\mathbf{p}^*, \mathbf{x}^*)\ (\mathbf{z}^*, \mathbf{v}^*) \geq (\mathbf{p}, \mathbf{x}^*)\ (\mathbf{z}^*, \mathbf{v}^*) \qquad [2]$$

and, for every $\mathbf{x} \in \mathbb{X}(k)$,

$$(\mathbf{p}^*, \mathbf{x}^*)\ (\mathbf{z}^*, \mathbf{v}^*) \geq (\mathbf{p}^*, \mathbf{x})\ (\mathbf{z}^*, \mathbf{v}^*) \qquad [3]$$

From [2] it follows that $\mathbf{p}^*\ \mathbf{z}^* \geq \mathbf{p}\ \mathbf{z}^*$, for all $\mathbf{p} \in \mathbb{P}$, which implies:

$$\mathbf{p}^*\ \mathbf{z}^* = \max_j.\ z_j^*$$

Therefore, the Walras Law and (A.2) imply that $\mathbf{p}^*\ \mathbf{z}^* = 0$, that is, $\max_j.\ z_j^* = 0$, and hence, $\mathbf{z}^* \leq \mathbf{0}$, with $p_s^* = 0$ whenever $z_s^* < 0$. Therefore, $\mathbf{x}^*$ corresponds to a feasible vector of activity levels.

Suppose now that a set of indices $J \neq \emptyset$ exists, such that

$$\mathbf{p}^*\ [B_j - (1 + \rho)A_j] > 0$$

for $j \in J$. Then, in view of [3], $\mathbf{x}^*$ must be a point in the set:

$$\{\ \mathbf{x} \in \mathbb{R}_+^h \ /\ \sum_{j=1}^{h} x_j = k\ \}$$

But this is not possible because, despite the fact that by construction this set only contains non-feasible activity levels, we have just seen that $\mathbf{x}^*$ is feasible. Therefore, $\mathbf{v}^* \leq \mathbf{0}$, and since $\mathbf{0} \in \mathbb{X}(k)$, we conclude:

$$\mathbf{p}^* \, [\mathbf{B} - (1 + \rho)\mathbf{A}] \, \mathbf{x}^* = 0$$

$\blacksquare$

Let $(\mathbf{p}^*, \mathbf{x}^*)$ be a solution to [1] (which exists, in view of Proposition 1). By construction this solution satisfies:

(i) $\hat{\xi}(\mathbf{p}^*, \mathbf{x}^*) \leq \mathbf{y}(\mathbf{x}^*)$

and

$$\hat{\xi}_s(\mathbf{p}^*, \mathbf{x}^*) < \sum_{j=1}^{h} y_{sj}(\mathbf{x}^*) \quad \Longrightarrow \quad p_s^* = 0, \; s = 1,2,\ldots, n+k$$

(ii) $\mathbf{p}^* \, [\mathbf{B} - (1 + \rho)\mathbf{A}] \leq \mathbf{0}$

and

$$\mathbf{p}^* \, [B_j - (1 + \rho)A_j] < 0 \quad \Longrightarrow \quad x_j^* = 0$$

Therefore, this solution will correspond to a competitive equilibrium relative to ρ if:

(a) Firms are maximizing profits at $(\mathbf{x}^*, \mathbf{p}^*)$.

(b) $\hat{\mathbf{d}}_i(\mathbf{p}^*, \mathbf{x}^*) = \mathbf{d}_i(\mathbf{p}^*)$, for all i.

Concerning point (a), observe that under assumption (A.1), firms maximize profits at $(\mathbf{x}^*, \mathbf{p}^*)$ turns out equivalent to saying that $\rho = \mathbf{r}(\mathbf{p}^*)$ (since otherwise competitive conditions imply that $\mu_g(\mathbf{p}^*) = 0$, and the *gth* firm's sole possibility is to produce nothing). But this is immediate. On the one hand, $\mathbf{x}^* \neq 0$ implies that ρ cannot exceed $\mathbf{r}(\mathbf{p}^*)$ (otherwise it would not be feasible). On the other, suppose that $\mathbf{r}(\mathbf{p}^*) > \rho$. Then it follows that:

$$\mathbf{p}^* \, (\mathbf{B} - [1 + \mathbf{r}(\mathbf{p}^*)]\mathbf{A}) << \mathbf{0}$$

so that $\mathbf{r}(\mathbf{p}^*) > \max_j \; r_j(\mathbf{p}^*)$, contradicting the definition of $\mathbf{r}(\mathbf{p}^*)$. Therefore, $\rho = r(\mathbf{p}^*)$ unless $\mathbf{x}^* = \mathbf{0}$.

Concerning point (b), observe now that for any given price vector $\mathbf{p}^* \in \mathbb{P}$, competitive conditions imply that:

(a) For all g, j, $x_{gj}(\mathbf{p}^*) > 0$ implies $r_j(\mathbf{p}^*) = \mathbf{r}(\mathbf{p}^*)$.

(b) For each $i = 1, 2, \ldots, m$, $\psi_{ig}(\mathbf{p}^*) = 0$ for all g such that

$$\sum_{j=1}^{h} \mathbf{p}^*[B_j - A_j]x_{gj}(\mathbf{p}^*) < \mathbf{r}(\mathbf{p}^*) \sum_{j=1}^{h} \mathbf{p}^*A_j x_{gj}(\mathbf{p}^*)$$

(c) $[\ \mathbf{r}(\mathbf{p}^*) > 0 \ \&\ \mathbf{x}^* > 0\ ] \implies \mathbf{p}^* \sum_{i=1}^{m} \sum_{g} \psi_{ig}(\mathbf{p}^*) = \mathbf{p}^* \, \omega$.

Property (a) says that no firm will operate a production process whose profitability be less than $r(\mathbf{p})$. This is so since technology is freely available and firms are assumed to be price takers.

Property (b) is the counterpart of (a). It establishes that no consumer will devote resources to those firms operating production processes whose profitability be less than the maximum attainable.

Property (c) says that each consumer will be willing to use all the resources with positive prices for productive purposes (this is equivalent to saying that the indirect utility function is monotone increasing in wealth, at given prices).

These properties imply that $\hat{\mathbf{d}}_i(\mathbf{p}^*, \mathbf{x}^*) = \mathbf{d}_i(\mathbf{p}^*)$, so that for each given ρ, a solution to [1] is actually a competitive equilibrium relative to ρ.

Finally, observe that once the equilibrium supply has been determined in the aggregate, the number of firms using each production process is undeterminated but irrelevant (in the sense that a continuum of possibilities exists).

Collecting all these results we get:

THEOREM .- Let E be an economy in $\mathbb{E}$. For any given $\rho \in \mathbb{R}$, there exists a Competitive Equilibrium relative to ρ.

5. COMMENTS AND REMARKS

We have presented a competitive market mechanism whose equilibria involve the equalization of firms' profitability. This mechanism combines the classical and neoclassical approaches to competitive markets, in the sense that equilibrium obtains *via* the simultaneous determination of prices and quantities (given consumers' preferences, endowments and technology) as a result of the interaction of price-taking agents seeking for the maximization of their objective functions.

There is a number of points worth commenting on, before concluding the paper.

1.- The structure of the model enables us to obtain some immediate conclusions. First notice that $\rho < 0$ implies $\mathbf{x}^* = \mathbf{0}$, since no consumer will contribute to the process of the firms creation (i.e., in this case $\mu_g(\mathbf{p}) = 0$, $\forall$ $\mathbf{p}$ in $\mathbb{P}$, $\forall$ g). It is also clear that there exists a positive value ρ' such that for all $\rho \geq \rho'$ the competitive equilibrium relative to ρ corresponds to a "pure exchange equilibrium" (this is so because technology is not productive enough to allow any process to yield such a rate of return).

The indecomposability of the production system [assumption (A.1)], the non-satiability of consumers [(iv) of assumption (A.2)], and the nature of competitive markets imply that when $\rho > 0$, every consumer will be willing to apply all her resources to production activities. Hence, if $x^* \neq 0$, $\delta_i(p^*) = 0$ for every i.

It is also interesting to note that, under assumption (A.1), the equalization of firms' profitability is a necessary condition for profit maximization at given prices, subject to the available inputs.

2.- As it happens with constant returns to scale economies, the equilibrium does not determine the number of firms. Hence, although it is not necessary for the existence result, it can be argued that a complete description of the proposed market mechanism is still required in order to comment on how coalitions are formed (meaning, by what principle the creation of firms may be governed). For that we may think of the auctioneer as calling not only a price vector, but also a *coalition structure* which is drawn at random. Each consumer may accept or decline to form part of the coalition(s) proposed by the auctioneer (at given prices). We may well assume that a consumer will actually agree to join her assigned coalition(s) if she cannot improve her position by joining some other(s).

According to this scheme, an equilibrium will define a industrial property structure, given by the (random) coalition structure associated to the equilibrium prices. The very definition of a competitive economy implies that in equilibrium all coalitions will be equally profitable.

3.- Our "Classical" General Equilibrium Model may be regarded as a mixture of those by von Neumann (1937), Debreu (1959) and Sraffa (1960) [see Bidard (1991, Chs. XXI and XXV) for a wider discussion]. Yet it departs from each of these reference models in several respects.

Unlike Debreu's (1959) general equilibrium model, we have allowed consumers to choose their participation in firms' property as part of their optimizing behaviour. Such behaviour does not assume that consumers will participate in a production economy for free. As a consequence, profit maximization implies the equalization of firms' profitability, and that there can be a positive equilibrium rate of profits under constant returns to scale. Observe that such a rate does not depend on the differences in commodity prices between two periods.

Unlike von Neumann's (1937) model of an expanding economy, and Sraffa's (1960) neo-Ricardian model of prices of production, we have introduced an explicit modeling of consumers' and firms' behaviour. The conflict between wages and profits here takes the form of a more general distributive problem: the equilibrium prices and rate of profits constitute the resultant of such *a multilateral "conflict" between the owners of resources.*

Let us remark that the model analyzed in this paper also inherits part of the shortcomings of the reference models. In particular: (a) The assumptions of price taking behaviour and complete information imply to leave unexplained the functioning of competitive markets; (b) The exogenous character of the rate of profits implies a strong indeterminacy of the model.

4.- Let us stress, finally, that in our model the rate of return is not to be interpreted as something related to time. Even though this interpretation is rather natural in many contexts, here it is more appropriate to think of it in a different way.

From the firms' point of view, the rate of return appears as a price they should pay for using something (wealth) which belongs to somebody else. The reason for which a "market" for such a product exists is that there may be something to earn if society engages in production activities. In this simplified world, the rate of return that consumers obtain from creating firms constitute a variable similar to the price one may get from allowing somebody else to join watching one's TV.

From a social perspective, production appears as a cooperative venture, in the sense that individuals (who own all means of production) have to transfer property rights to collective institutions (the firms) in order to produce a joint outcome. Then, the rate of return may be thought of as a parameter which tells us how to share the benefits from this cooperative outcome.

REFERENCES

Arrow, K. J. & Hahn, F.H. (1971) **General Competitive Analysis**, Holden Day, San Francisco.

Bidard, Ch., (1991), **Prix, Reproduction, Rareté**, Dunod, Paris.

Debreu, G. (1959), **Theory of Value. An Axiomatic Analysis of Economic Equilibrium**, Cowles Foundation Monograph n° 17, John Wiley, New York.

von Neumann, J. (1937), Uber ein Ökonomisches Gleichungssytem und eine Verallgemeinerung des Brouwerschen Fixpunktsatzes, **Ergebnisse eines Mathematischen Kolloquiums**, 8 (translated as "A Model of General Equilibrium", **Review of Economic Studies**, 13 : 1-9, 1945-46).

Sraffa, P. (1960), **Production of Commodities by Means of Commodities**, Cambridge University Press, Cambridge.

Villar, A. (1991), **Operator Theorems with Applications to Distributive Problems and Equilibrium Models**, Springer-Verlag, forthcoming.

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