**MARKET EQUILIBRIUM WITH NONCONVEX TECHNOLOGIES***

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A B S T R A C T

An economy with $\ell$ commodities, $m$ consumers and $n$ firms is considered. Consumers are modelled in a standard way. It is assumed that the *jth* firm has a closed and comprehensive production set, $Y_j$, with $0 \in Y_j$. The equilibrium of firms appears associated to the notion of a *pricing rule* (a mapping applying the boundary of a firm's production set on the price space, whose graph describes the pairs prices-production which a firm finds "acceptable"). We show that when firms follow loss-free and upper hemicontinuous, convex-valued pricing rules, a price vector and an allocation exist, such that: a) Consumers maximize their preferences subject to their budget constraints; b) Every firm is in equilibrium; and c) All markets clear.

## I.- INTRODUCTION.

The aim of this paper is to provide some sufficient conditions for the existence of market equilibria in economies allowing for non-convex production sets, such that different firms can follow different policies.

It is well known that general equilibrium models face serious difficulties in the presence of non-convex technologies, when there are finitely many firms. Such difficulties are both analytical and theoretical and have mainly to do with the fact that the supply correspondence may not be continuous, convex-valued or even defined, so that (non-cooperative Nash) equilibrium typically fails to exist. Alternative techniques of analysis and different equilibrium concepts must be applied [see Mas-Colell (1987), Cornet (1988) and Dehez (1988), for a review of the different lines of research to which these problems give rise to].

The model developed in this paper refers to those economies which may exhibit non-convex technologies. The approach we follow is related to that literature which considers extensions of the standard Arrow-Debreu-McKenzie general equilibrium model, allowing for the presence of increasing returns to scale [see for instance Böhm (1986), Scarf (1986), Dehez & Drze (1988 a, b), Kamiya (1988), Vohra (1988), and specially Bonnisseau & Cornet (1988)].

Common to all these models is the idea that an equilibrium may be understood as a price vector, a list of consumption allocations and a list of production plans such that: (a) all agents face the same prices; (b) the markets for all goods clear; (c) the consumers maximize their preferences subject to their budget constraints; and (d) each individual firm is in "equilibrium" at those prices and production plans. It is mostly the nature of the equilibrium condition (d) what establishes the differences between these models (both with respect to each other and with respect to the Walrasian one).

A distinctive feature of these contributions is that the equilibrium of firms appears associated to the notion of a *pricing rule*, rather than to that of a supply correspondence. A *pricing rule* is a mapping applying the boundary of a firm's production set on the price space. The graph of such a mapping describes the pairs prices-production which a firm finds "acceptable" (a pricing rule may be thought of as the inverse mapping of a generalized supply correspondence). These mappings may be continuous and convex-valued even when the supply correspondence is not. Prominent examples of these pricing rules can be seen in profit maximization (under convex technologies), average or marginal cost pricing, or voluntary trading.

Our model here follows these lines. We shall consider a private ownership market economy with only two types of agents: Consumers and firms. Consumers will be modelled in a standard way [as in Debreu (1959, Ch. 4), say]. Concerning the production side, it is assumed that the *jth* firm has a closed and comprehensive production set, $Y_j$, with $0 \in Y_j$, and an upper hemicontinuous, and convex valued pricing rule, $\Phi_j$. A firm is said to be in equilibrium when $(\mathbf{y}_j, \mathbf{p}) \in Gr.\Phi_j$.

A specific target of our analysis is to separate the problems created by the existence of nonconvexities in order to define the behaviour of firms, from those derived from normative considerations (e.g., marginal cost pricing), or consumers' minimal wealth requirements. This limit of scope clarifies the presentation of the model without actually limiting its applicability. Thus we shall only be concerned with markets where firms can remain inactive at no cost, and with loss-free pricing rules. The term "Market Equilibrium" alludes to these features.

The main result of this work shows that when firms follow loss-free pricing rules, an equilibrium exists (that is, a price vector and an allocation exist, such that: a) Consumers maximize their preferences subject to their budget constraints; b) Every firm is in equilibrium; and c) All markets clear).

The strategy of the proof involves transforming the equilibrium problem into a variational inequalities problem. This requires the construction of a suitable mapping defined on a convex set which is homeomorphic to the cartesian product of the efficient production sets.

The model in this paper may be thought of as a particular case of that one analyzed in Bonnisseau & Cornet (1988, Th. 2.1'). In this respect our contribution consists of presenting an easier-to-handle model and an alternative proof of the existence theorem (an alternative proof which we find simpler and more intuitive).

The rest of the paper is organized as follows. Section II presents the model, whilst Section III develops the proof of the existence Theorem. A few final remarks conclude the work.

## II.- THE MODEL.

Consider a market economy with $\ell$ perfectly divisible commodities and a given number of economic agents which can be either consumers (with cardinal *m*) or firms (with cardinal *n*). A point $\omega \in \mathbb{R}^{\ell}$ denotes the vector of initial endowments.

Following the standard convention, the technological possibilities of the *jth* firm (j = 1, 2, ..., n) are represented by a subset $Y_j$ of $\mathbb{R}^{\ell}$ (to be referred to as the *jth* firm production set). We shall denote by $\mathfrak{F}_j$ the *jth* firm's set of (weakly) efficient production plans, that is,

$$\mathfrak{F}_j \equiv \{ \mathbf{y}_j \in Y_j \ / \ \mathbf{y}'_j >> \mathbf{y}_j \implies \mathbf{y}'_j \notin Y_j \}$$

$\mathfrak{F}$ will stand for the cartesian product of the *n* sets of (weakly) efficient production plans, that is,

$$\mathfrak{F} \equiv \prod_{j=1}^{n} \mathfrak{F}_j$$

We shall denote by $\mathbb{P} \subset \mathbb{R}^{\ell}_{+}$ the standard price simplex, that is,

$$\mathbb{P} = \{ \mathbf{p} \in \mathbb{R}^{\ell}_{+} \ / \ \sum_{t=1}^{\ell} p_t = 1 \}$$

For a point $\mathbf{y}_j \in \mathfrak{F}_j$ and a price vector $\mathbf{p} \in \mathbb{P}$, $\mathbf{p}\ \mathbf{y}_j$ gives us the associated profits.

Each firms' behaviour is defined in terms of a *Pricing Rule*. A Princing Rule for the *jth* firm is usually defined as a mapping $\Phi_j$ applying the set of efficient production plans, $\mathfrak{F}_j$ into $\mathbb{R}^{\ell}_{+}$. For a point $\mathbf{y}_j \in \mathfrak{F}_j$,

$\Phi_j(\mathbf{y}_j)$ has to be interpreted as the set of price vectors the *jth* firm finds "acceptable" when producing $\mathbf{y}_j$. In other words, the *jth* firm is in equilibrium at the pair $(\mathbf{y}_j, \mathbf{p})$, if $\mathbf{p} \in \Phi_j(\mathbf{y}_j)$.

We shall adopt a more general notion of firms' behaviour, by allowing for each firm's Pricing Rule to depend on other firms actions and "market prices". For that, let

$$\bar{\mathbf{y}} = (\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n)$$

denote a point in $\mathfrak{F}$. Then,

Definition.- A **Pricing Rule for the jth firm** is a correspondence,

$$\phi_j : \mathbb{P} \times \mathfrak{F} \dashrightarrow \mathbb{P}$$

which establishes the *jth* firm set of admissible prices, as a function of "market conditions". That is, $\mathbf{y}_j$ is an equilibrium production plan for the *jth* firm at prices $\mathbf{p}$, if and only if, $\mathbf{p}$ belongs to $\phi_j(\mathbf{p}, \bar{\mathbf{y}})$ (where $\mathbf{y}_j$ is precisely the *jth* firm production plan in $\bar{\mathbf{y}}$).

As for interpretative purposes, we may think of a market mechanism in which there is an auctioneer who calls out both a price vector (to be seen as proposed market prices), and a vector of efficient production plans. Then, the *jth* firm checks out whether the pair $(\mathbf{p}, \mathbf{y}_j)$ agrees with her objectives (formally, $[(\mathbf{p}, \bar{\mathbf{y}}), \mathbf{p}]$ belongs to the graph of $\phi_j$).

When $\mathbf{p} \in \bigcap_{j=1}^{n} \phi_j(\mathbf{p}, \bar{\mathbf{y}})$, then $\mathbf{p}$ is candidate for a market equilibrium (usually called a "Production Equilibrium").

**Remark.-** Observe that different firms may follow different pricing rules. Furthermore, the pricing rule "may be either endogenous or exogenous to the model, and that it allows both price-taking and price-setting behaviours" [Cf. Cornet (1988, p. 106)].

There are *m* consumers. Each consumer i = 1,2,..., m, is characterized by a tuple,

$$[C_i,\ u_i,\ \omega_i,\ \{\ \theta_{ij}\ \}\ ]\quad ,$$

where $C_i$, $u_i$ and $\omega_i$ stand for the *ith* consumer consumption set, utility function and initial endowments, respectively, and $\theta_{ij}$ denotes the *ith* consumer's participation in the *jth* firm's profits. By definition, $\sum_{i=1}^{m} \omega_i = \omega$ , and $\theta_{ij} \geq 0$ for all *i,j*, with $\sum_{i=1}^{m} \theta_{ij} = 1$, for all *j*. We shall follow the convention of denoting with negative numbers those commodities a consumer may supply.

Given a price vector **p**, and a vector of production plans $(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_n) \in \mathfrak{Y}$, the *ith* consumer's behaviour is obtained from solving the following program:

$$\text{Max.}\quad u_i(c_i)$$

$$\text{s.t.}:$$

$$c_i \in C_i$$

$$\mathbf{p}\ c_i \leq \mathbf{p}\ \omega_i + \sum_{j=1}^{n} \theta_{ij}\ \mathbf{p}\ \mathbf{y}_j$$

Let $\bar{y} = (y_1, y_2, \ldots, y_n)$ denote a point in $\mathfrak{Y}$. Then, consumers' behaviour can be summarized by an aggregate net demand correspondence, that can be written as follows:

$$\xi(p, \bar{y}) = d(p, \bar{y}) - \{ \omega \}$$

where $d(p, \bar{y}) \equiv \sum_{i=1}^{m} d_i(p, \bar{y})$, and $d_i(p, \bar{y})$ stands for the set of solutions to the program above for $(p, \bar{y})$.

For a given vector of initial endowments, $\omega \in \mathbb{R}^{\ell}$, let $\mathcal{A}(\omega)$ denote the set of attainable allocations, that is, $\mathcal{A}(\omega)$ consists of points

$$[(c_i), \bar{y}] \in \prod_{i=1}^{m} C_i \times \prod_{j=1}^{n} Y_j$$

such that $\sum_{i=1}^{m} c_i - \omega \leq \sum_{j=1}^{n} y_j$. The projection of $\mathcal{A}(\omega)$ on $Y_j$ (resp. $C_i$) defines the *jth* firm set of attainable productions (resp. the *ith* consumer set of attainable consumptions).

Consider now the following assumptions:

A.1.- For each firm $j = 1, 2, \ldots, n$,

(i) $Y_j$ is a closed subset of $\mathbb{R}^{\ell}$.

(ii) $Y_j \cap \mathbb{R}^{\ell}_+ = \{ 0 \}$.

(iii) $Y_j - \mathbb{R}^{\ell}_+ \subset Y_j$.

(iv) For each given vector of initial endowments, $\omega \in \mathbb{R}^{\ell}$, the *jth* firm set of attainable productions is bounded.

**A.2.-** For each $i = 1, 2, \ldots, m$,

(i) $C_i$ is a closed and convex subset of $\mathbb{R}^{\ell}$, bounded from below.

(ii) $u_i : C_i \rightarrow \mathbb{R}$ is a continuous and quasi-concave function.

(iii) $\omega_i \in$ int.$C_i$.

(iv) (Local non-satiation) For each $c_i \in C_i$, and for every $\varepsilon > 0$, there exists $c'_i \in B(c_i, \varepsilon) \cap C_i$ such that $u_i(c'_i) > u_i(c_i)$ (where $B(c_i, \varepsilon)$ stands for a closed ball with centre $c_i$ and radius $\varepsilon$).

Assumption (A.1) is common to those models where firms can remain inactive. Besides the technical point (i), point (ii) explicitly assumes that firms cannot produce without using up some inputs, and that $0 \in Y_j$ (which ensures that $Y_j$ is nonempty). Point (iii) corresponds to the free-disposal assumption. Finally, point (iv) says that it is not possible for the *jth* firm to obtain unlimited productions out of a (finite) given vector of initial endowments. Observe that under (A.1) the set of weakly efficient production plans, $\mathfrak{F}_j$, consists exactly of those points in the boundary of $Y_j$.

Assumption (A.2) is standard and needs no comment [see Debreu (1962) for some relaxations of these hypotheses].

**Remark.-** Observe that we have defined the *jth* firm pricing rule as a mapping $\phi_j$ applying $\mathbb{P} \times \mathfrak{F}$ into $\mathbb{P}$, rather than into $\mathbb{R}^\ell$ (which would have been a more general setting). Yet, it is clear that under assumptions (A.1) and (A.2) there is no loss of generality in such a definition.

Let us now introduce the definitions which make explicit the key concepts leading up to our main result.

Definition.- We shall say that $\phi_j : \mathbb{P} \times \mathfrak{Y} \dashrightarrow \mathbb{P}$ is a **Loss-Free Pricing Rule,** if for each $(\mathbf{p}, \bar{\mathbf{y}})$ in $\mathbb{P} \times \mathfrak{Y}$, all $\mathbf{q}_j$ in $\phi_j(\mathbf{p}, \bar{\mathbf{y}})$, we have:

$$\mathbf{q}_j \, \mathbf{y}_j \geq 0$$

Definition.- We shall say that $\phi_j : \mathbb{P} \times \mathfrak{Y} \dashrightarrow \mathbb{P}$ is a **Regular Pricing Rule,** if $\phi_j$ is an upper hemicontinuous correspondence, with nonempty, closed and convex values.

**Remark.-** The combination of the notions of loss-freeness and regularity imply a specific structure on $\phi_j$ as it approaches to $\mathbf{y}_j = \mathbf{0}$. In particular, it prevents for a firm to set

$$\phi_j(\mathbf{p}, \bar{\mathbf{y}}) \equiv \{ \mathbf{q}^o \}$$

(constant) for all $(\mathbf{p}, \bar{\mathbf{y}})$ in $\mathbb{P} \times \mathfrak{Y}$ (which would easily destroy any possibility of equilibria). The reader is encouraged to think about the nature of this implication [Bonnisseau & Cornet (1988, Remark 2.6) will help].

Definition.- We shall say that a price vector $\mathbf{p}^* \in \mathbb{P}$, and an allocation $[(c_i^*), \bar{\mathbf{y}}^*]$, yield a **Market Equilibrium** if the following conditions are satisfied:

($\alpha$) For each $i = 1, 2, \ldots, m$, $c_i^*$ maximizes $u_i$ over the set of points $c_i$ in $C_i$ such that:

$$\mathbf{p}\, c_i \leq \mathbf{p}^* \omega_i + \sum_{j=1}^{n} \theta_{ij} \; \mathbf{p}^* \, \mathbf{y}_j^*$$

($\beta$) For every $j = 1, 2, \ldots, n$, the *jth* firm is in equilibrium, that is,

$$p^* \in \bigcap_{j=1}^{n} \phi_j(p^*, \bar{y}^*)$$

$(\gamma)$ $\sum_{i=1}^{m} c_i^* - \sum_{j=1}^{n} y_j^* \leq \omega$, and:

$$\sum_{i=1}^{m} c_{it}^* - \sum_{j=1}^{n} y_{jt}^* < \omega_t \implies p_t^* = 0$$

That is, a Market Equilibrium is a situation in which: (a) Consumers maximize their preferences subject to their budget constraints; (b) Every firm is in equilibrium; and (c) All markets clear.

Let $\mathbb{E}$ denote the class of economies just described, that is, private ownership market economies satisfying assumptions (A.1) and (A.2). The main result of the paper is the following:

**THEOREM.- Let E stand for an economy in $\mathbb{E}$. A Market Equilibrium exists when firms follow regular and loss free pricing rules.**

(The proof is given in Section III).

This Theorem says that for the class of economies defined by assumptions (A.1) and (A.2), the regularity and loss-freeness of firms' pricing rules, constitute sufficient conditions for the existence of market equilibria.

**Remark.-** Observe that since consumers' choices depend on market prices and firms' production, we may think of each $\phi_j$ as also being dependent on consumers' decisions, that is,

$$\phi_j(p, \bar{y}) = \Theta_j[p, \bar{y}, \xi(p, \bar{y})].$$

This provides enough flexibility to deal with market situations in which firms' target payoffs may depend on demand conditions.

Let us consider now three cases which provide prominent examples of regular and loss-free pricing rules: Profit maximization, average cost pricing and voluntary trading [see Bonnisseau & Cornet (1988, Section 3), and Dehez & Drze (1988 a, Lemma 1)].

(A) Profit maximization (under convex technologies)

When technologies are convex, profit maximization can be defined in terms of the following pricing rule:

$$\phi_j^{PM}(p, \bar{y}) \equiv \{ q \in \mathbb{R}_+^{\ell} \ / \ q\, y_j \geq q\, y'_j \ , \ \forall\, y'_j \in Y_j \}$$

Under assumption (A.1), this is obviously a *loss-free* pricing rule (since $0 \in Y_j$ for each $j$); it is also easy to deduce that $\phi_j^{PM}$ is *regular* (the maximum theorem implies the upper hemicontinuity, whilst the convexity of $Y_j$ brings about the convex valuedness). *Thus, in particular, the existence of a Walrasian equilibrium is obtained as a Corollary of this Theorem.*

(B) Average cost-pricing

Average cost-pricing is a pricing rule with a long tradition in economics (both in positive and normative analysis). It can be formulated as follows[1]:

$$\phi_j^{AC}(p, \bar{y}) \equiv \{ q \in \mathbb{R}_+^{\ell} \ / \ q\, y_j = 0 \}$$

Under assumption (A.1), $\phi_j^{AC}$ is a loss-free and regular pricing rule (by reasonsing as above). Hence, the Theorem provides an implicit existence result for those economies where firms are instructed to get zero profits.

Average cost-pricing belongs to a family of loss-free and regular pricing rules whose associated equilibria may be difficult to sustain, since some firms may find it profitable to deviate from the equilibrium production plans (think of decreasing returns to scale). The next pricing rule intends to overcome such a difficulty, by requiring cost minimization [the reader may well consult Scarf (1986) and Dehez & Drze (1988 b) for a discussion of this type of problem].

(C) Voluntary trading

Dehez & Drze (1988 a) introduce the notion of *voluntary trading* as a way of extending the notion of competitive equilibria to a context where

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This expression corresponds to the case where $y_j \neq 0$. For $\bar{y}$ such that $y_j = 0$, the *jth* firm pricing rule must be defined as the closed convex hull of the following set:

$$\{ q \in \mathbb{R}_+^{\ell} \ / \ \exists \{q^{\nu}, y_j^{\nu}\} \subset \mathbb{R}_+^{\ell} \times [\mathfrak{Y}_j \setminus 0] \text{ , such that, } \{q^{\nu}, y_j^{\nu}\} \longrightarrow (q, 0), \text{ with } q^{\nu} y_j^{\nu} = 0 \}$$

firms behave as quantity takers, and there may be increasing returns to scale. This pricing rule is defined as follows:

$$\phi_j^{VT}(\mathbf{p}, \bar{\mathbf{y}}) \equiv \{ \mathbf{q} \in \mathbb{R}_+^{\ell} \ / \ \mathbf{q}\, \mathbf{y}_j \geq \mathbf{q}\, \mathbf{y}, \ \ \forall\, \mathbf{y} \in Y_j \ \text{ with } \mathbf{y} \leq \mathbf{y}_j^+ \ \}$$

(where $\mathbf{y}_j^+$ denotes a vector in $\mathbb{R}_+^{\ell}$ with coordinates max. $\{ 0, y_{jh} \}$, for $h = 1, 2, \ldots, \ell$). They show that $\phi_j^{VT}$ is a loss-free and regular pricing rule which collapses to profit maximization under convex technologies. Our Theorem provides an equilibrium existence result based on a proof similar to theirs.

## III.- THE EXISTENCE OF MARKET EQUILIBRIA.

Let $\mathbb{E}$ denote the class of private ownership market economies satisfying assumptions (A.1) and (A.2). We shall show that these assumptions suffice in order to prove the existence of Market Equilibria for regular and loss-free pricing rules.

The following Lemma will be the key for the existence result below:

**Lemma 1.- Let D be a compact and convex subset of $\mathbb{R}^{\ell}$, and $\Gamma$:D --> $\mathbb{R}^{\ell}$ an upper-hemicontinuous correspondence, with nonempty, compact and convex values. Then points $x^* \in D$, $y^* \in \Gamma(x^*)$ exist such that,**

$$(x - x^*)\, y^* \leq 0 \quad ,$$

**for all $x \in D$.**

Proof.-

Let $T = \Gamma(D)$. Since D is compact, T will be a compact set. Let Co(T) denote the convex hull of T. By construction Co(T) is a compact and convex set. Now define a correspondence $\mu$:Co(T) ---> D as follows:

$$\mu(y) = \{\ x \in D \ / \ x\, y \geq z\, y\ ,\ \forall\ z \in D\ \}$$

Clearly $\mu$ is a nonempty, convex-valued correspondence. Furthermore, $\mu$ is upper hemicontinuous.

Define now a new correspondence, $\pi$ from DxCo(T) into itself as follows:

$$\pi(x,\ y) = \mu(y) \times \Gamma(x)$$

By construction, $\pi$ is an upper-hemicontinuous correspondence with nonempty, compact and convex values, applying a compact and convex set into itself. Thus, Kakutani's Fixed Point Theorem applies, and there exists $(x^*, y^*) \in \pi(x^*, y^*)$, that is,

$$x^* \in \mu(y^*), \; y^* \in \Gamma(x^*)$$

By definition of $\mu$ we have:

$$x^* y^* = \max.\ z\, y^* \text{ , for all } z \text{ in } D$$

and hence the result follows.

■

For each j = 1, 2, ..., n, define a mapping $g_j : \mathbb{P} \times \mathfrak{Y} \dashrightarrow \mathbb{R}^\ell$ such that it associates to every $(\mathbf{p}, \bar{\mathbf{y}})$ in $\mathbb{P} \times \mathfrak{Y}$ the set of points $\mathbf{t}_j$ which solve the following program:

$$\begin{aligned} &\underset{\mathbf{t}_j}{\text{Min.}} \;\; \text{dist.}[\, \mathbf{t}_j, \mathbf{y}_j \,] \\ \text{s.t.:}\; & \\ &\mathbf{p}\, \mathbf{t}_j \geq 0 \end{aligned}$$

It is easy to check that, for each $(\mathbf{p}, \bar{\mathbf{y}})$ in $\mathbb{P} \times \mathfrak{Y}$ , there is a unique solution to this program, which varies continuously with $(\mathbf{p}, \bar{\mathbf{y}})$. Thus, for each *j*, $g_j$ is actually a continuous function[2]. We shall write $g(\mathbf{p}, \bar{\mathbf{y}}) \equiv \sum_{j=1}^{n} g_j(\mathbf{p}, \bar{\mathbf{y}})$, which obviously is a continuous function of its arguments.

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[2] Observe that when prices are such that the *jth* firm has nonnegative profits, $g_j(\mathbf{p}, \bar{\mathbf{y}})$ coincides with the corresponding net production. For those points satisfying $\mathbf{p}\, \mathbf{y}_j < 0$, $g_j(\mathbf{p}, \bar{\mathbf{y}})$ is the unique point in the hyperplane $\mathbf{p}\, \mathbf{z} = 0$ at minimum distance of $\mathbf{y}_j$.

Consider now the following set:

$$\mathcal{A}'(\omega) \equiv \{ [(c_i)\ (y_j)] \in \prod_{i=1}^{m} C_i \times \prod_{j=1}^{n} \mathfrak{Y}_j \ / \ \sum_{i=1}^{m} c_i - \omega \leq \sum_{j=1}^{n} g_j(p, \bar{y}) \},\ \forall\ p \in \mathbb{P} \}$$

which is a nonempty and compact subset of $\mathbb{R}^{\ell(m+n)}$, under assumptions (A.1) and (A.2). Let $\mathfrak{Y}'_j$ denote the projection of $\mathcal{A}'(\omega)$ on $\mathfrak{Y}_j$, with $\mathfrak{Y}' \equiv \prod_{j=1}^{n} \mathfrak{Y}'_j$. Let now $\mathfrak{Y}^*_j$ stand for a compact subset of $\mathfrak{Y}_j$ such that $\mathfrak{Y}'_j \subset \text{int}.\mathfrak{Y}^*_j$, and let $\mathfrak{Y}^* \equiv \prod_{j=1}^{n} \mathfrak{Y}^*_j$.

By construction, for each $(p, \bar{y})$ in $\mathbb{P} \times \mathfrak{Y}^*$, $p\ g_j(p, \bar{y}) \geq 0$.

The following Lemma is a direct consequence of assumption (A.1), and hence the proof will be omitted [see Bonnisseau & Cornet (1988, Lemma 5.1)].

**Lemma 2.- Let $Y_j$ be a production set satisfying (A.1), and let $\mathfrak{Y}^*_j$ stand for a compact subset of $\mathfrak{Y}_j$, such that $\mathfrak{Y}'_j \subset \text{int}.\mathfrak{Y}^*_j$. Then, $\mathfrak{Y}^*_j$ can be made homeomorphic to a simplex:**

$$X_j = \{ x_j \in \mathbb{R}^{\ell}_{+} \ / \ \sum_{i=1}^{\ell} x_{ij} = 1 \}$$

**so that the set of points in $\mathfrak{Y}'_j$ are mapped into the interior of $X_j$.**

For each $j = 1, 2, \ldots, n$, let $h_j$ denote the (continuous) inverse mapping which associates to every $\mathbf{x}_j$ in $X_j$ a unique $\mathbf{y}_j$ in $\mathfrak{Y}^*_j$. Consider now the following sets:

$$\mathbb{X} \equiv \prod_{j=1}^{n} X_j$$

$$\Delta \equiv \mathbb{P} \times \mathbb{X}$$

$\bar{\mathbf{x}} = (\mathbf{x}_1, \ldots, \mathbf{x}_n)$ will denote a point in $\mathbb{X}$. We shall write:

$$\hat{g}_j(\mathbf{p}, \bar{\mathbf{x}}) \equiv g_j[\mathbf{p}, h_j(\mathbf{x}_j)]$$

$$\hat{g}(\mathbf{p}, \bar{\mathbf{x}}) \equiv \sum_{j=1}^{n} \hat{g}_j(\mathbf{p}, \bar{\mathbf{x}})$$

which are obviously continuous functions on $\Delta$.

Let now $\hat{E}$ denote an economy identical to E in all respects but consumers' budget sets. More precisely, define

$$\hat{\xi}(\mathbf{p},\bar{\mathbf{x}}) \equiv \hat{\mathbf{d}}(\mathbf{p}, \bar{\mathbf{x}}) - \{ \omega \} \quad ,$$

where:

$$\hat{d}(\mathbf{p}, \bar{\mathbf{x}}) \equiv \sum_{i=1}^{m} \hat{d}_i(\mathbf{p}, \bar{\mathbf{x}}),$$

and $\hat{d}_i(\mathbf{p}, \bar{\mathbf{x}})$ stands for the set of solutions to the program:

$$\text{Max. } u_i(c_i)$$

s.t. :

$$c_i \in C_i$$

$$\mathbf{p}\, c_i \le \mathbf{p}\, \omega_i + \sum_{j=1}^{n} \theta_{ij}\, \mathbf{p}\, \hat{g}_j(\mathbf{p}, \bar{\mathbf{x}})$$

Let $h(\bar{\mathbf{x}}) \equiv [h_1(\mathbf{x}_1), h_2(\mathbf{x}_2), \ldots, h_n(\mathbf{x}_n)]$, and define a mapping $\Gamma: \Delta \dashrightarrow \mathbb{R}^{\ell(1+n)}$ as follows:

$$\Gamma(p, \bar{x}) \equiv \begin{bmatrix} \hat{\xi}(p, \bar{x}) - \hat{g}(p, \bar{x}) \\ p - \phi_1[p, h(\bar{x})] \\ p - \phi_2[p, h(\bar{x})] \\ \vdots \\ p - \phi_n[p, h(\bar{x})] \end{bmatrix}$$

We are ready now to present our main result:

**THEOREM .- Let E stand for an economy in $\mathbb{E}$. A Market Equilibrium exists when firms follow regular and loss free pricing rules.**

Proof.-

Let $\hat{E}$ stand for the economy defined above. Observe that assumption (A.2), and the definition of $\hat{g}_j$ imply that for all $i$ = 1,2,..., m, the mapping $\gamma_i:\Delta \to C_i$ given by:

$$\gamma_i(p, \bar{x}) = \{ c_i \in C_i \ / \ p\, c_i \le p\, \omega_i + \sum_{j=1}^{n} \theta_{ij}\, p\, \hat{g}_j(p, \bar{x}) \}$$

is continuous in $(p, \bar{x})$ [Debreu (1959, 4.8 (1) )], with nonempty, compact and convex values. Therefore, since preferences are assumed to be continuous and convex, for each pair $(p, \bar{x}) \in \Delta$, every $\hat{d}_i$ will be an upper-hemicontinuous correspondence (the maximum theorem applies here), with nonempty, compact and convex values. Consequently, $\hat{\xi}$ inherits these properties and, by hypothesis, and in view of the Lemma, this also applies for $\Gamma$.

Thus, $\Gamma$ is an upper-hemicontinuous correspondence, with nonempty, compact and convex values, applying a compact and convex set, $\Delta \subset \mathbb{R}_+^{\ell(1+n)}$

into $\mathbb{R}^{\ell(1+n)}$. Then, Lemma 1 ensures the existence of points $(\mathbf{p}^*, \bar{\mathbf{x}}^*) \in \Delta$, $(\mathbf{z}^*, \mathbf{v}^*) \in \Gamma(\mathbf{p}^*, \bar{\mathbf{x}}^*)$ such that,

$$(\mathbf{p}^*, \bar{\mathbf{x}}^*)\ (\mathbf{z}^*, \mathbf{v}^*) \geq (\mathbf{p}, \bar{\mathbf{x}})\ (\mathbf{z}^*, \mathbf{v}^*)$$

for every pair $(\mathbf{p}, \bar{\mathbf{x}})$ in $\Delta$. In particular,

$$(\mathbf{p}^*, \bar{\mathbf{x}}^*)\ (\mathbf{z}^*, \mathbf{v}^*) \geq (\mathbf{p}, \bar{\mathbf{x}}^*)\ (\mathbf{z}^*, \mathbf{v}^*) \quad , \forall\ \mathbf{p} \in \mathbb{P} \qquad [1]$$

$$(\mathbf{p}^*, \bar{\mathbf{x}}^*)\ (\mathbf{z}^*, \mathbf{v}^*) \geq (\mathbf{p}^*, \bar{\mathbf{x}})\ (\mathbf{z}^*, \mathbf{v}^*) \quad , \forall\ \bar{\mathbf{x}} \in \mathbb{X} \qquad [2]$$

From [1] it follows that $\mathbf{p}^*\ \mathbf{z}^* \geq \mathbf{p}\ \mathbf{z}^*$, for all $\mathbf{p} \in \mathbb{P}$, which implies:

$$\mathbf{p}^*\ \mathbf{z}^* = \max_j.\ z_j^*$$

Therefore, the Walras Law implies that $\mathbf{p}^*\ \mathbf{z}^* = 0$, that is,

$$\max_j.\ z_j^* = 0\ ,$$

and hence, $\mathbf{z}^* \leq \mathbf{0}$ [that is, this allocation belongs to $\mathcal{A}'(\omega)$], with $p_s^* = 0$ whenever $z_s^* < 0$.

Similarly, from [2] it follows that $\bar{\mathbf{x}}^*\ \mathbf{v}^* \geq \bar{\mathbf{x}}\ \mathbf{v}^*$, for all $\bar{\mathbf{x}}$ in $\mathbb{X}$, and hence for each $j$ we get: $\mathbf{x}_j^*\ \mathbf{v}_j^* \geq \mathbf{x}_j\ \mathbf{v}_j^*$ for all $\mathbf{x}_j$ in $X_j$. Consequently, for each $j$,

$$\mathbf{x}_j^*\ \mathbf{v}_j^* = \max_i.\ v_{ij}^*$$

which implies that either $v_{kj}^* = \{\max_i.\ v_{ij}^*\}$, or $x_{kj}^* = 0$. Since $\mathbf{z}^* \leq 0$ implies that the corresponding allocation belongs to $\mathcal{A}'(\omega)$, by construction, each $\mathbf{x}_j^*$ must be a point in the interior of $X_j$, that is, $v_{kj}^* = \{\max_i.\ v_{ij}^*\}\ \forall\ k,\ \forall\ j$. Now, since for each $j$, $\mathbf{v}_j^* \equiv \mathbf{p}^* - \mathbf{q}_j^*$ consists of the difference between two points in $\mathbb{P}$ (for some $\mathbf{q}_j^* \in \phi_j[\mathbf{p}^*, h(\bar{\mathbf{x}}^*)]$), it must be the case that $\mathbf{v}_j^* = 0$ for all $j$, that is,

$$\mathbf{p}^* \in \bigcap_{j=1}^{n} \phi_j[\mathbf{p}^*, h(\mathbf{x}^*)]$$

Therefore, there exists a market equilibrium for $\hat{E}$.

Finally, since $\phi_j$ is a loss-free pricing rule, for all $j$, it follows that:

$$\mathbf{p}^* \, h_j(\mathbf{x}_j^*) \geq 0$$

Therefore, $\hat{g}_j(\mathbf{p}^*, \bar{\mathbf{x}}^*) = h_j(\mathbf{x}_j^*)$, and consequently

$$0 \geq \mathbf{z}^* \in \xi(\mathbf{p}^*, \bar{\mathbf{x}}^*) - \sum_{j=1}^{n} h_j(\mathbf{x}_j^*)$$

that is, $\mathbf{p}^*$ yields actually a Market Equilibrium for E.

That completes the proof.

$\blacksquare$

## IV.- SUMMARY AND CONCLUSIONS.

A model of a private ownership market economy allowing for non-convex technologies, such that different firms may follow different policies, has been analyzed in this paper. An equilibrium existence theorem has served to ensure its logical consistency.

The model has been focused on economic environments involving only two types of agents (consumers and firms), such that firms can remain inactive at no cost. In so doing we have tried to separate those problems created by the existence of nonconvexities in order to define the behaviour of firms, from those derived from normative considerations or consumers's minimal wealth requirements. Yet, the proof of the Theorem makes it clear that there is no difficulty in allowing for more general profit rules, the only relevant restriction being the preservation of the upper hemicontinuity of $\hat{\xi}$.

Firms' equilibrium has been linked to the notion of a *pricing rule*. A pricing rule describes how firms' behaviour is related to market conditions. The graph of such a mapping tells us the combinations of production plans and prices, that the firms find "acceptable". This approach allows for a wide range of behaviour, and becomes operational under standard conditions (regularity).

A market equilibrium is a situation in which all agents are in equilibrium and all markets clear.

Finally, let us mention that even though the model we present here clearly belongs to the family of extensions of the Arrow-Debreu-MacKenzie general equilibrium model, it was arrived at along a different line of research. Indeed, this paper originates as a further step in a series of extensions of nonlinear Leontief models, as those in Corch"n (1988) or Herrero & Villar (1988).

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